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THE REDUCIBILITY OF THE EQUATIONS OF FREE MOTION OF A COMPLEX MECHANICAL SYSTEM†

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A mechanical system consists of an unchangeable rigid body (a carrier) and a subsystem whose configuration and composition may vary with time (the motion of its elements relative to the carrier is given). The free motion of the system in a uniform gravitational field is investigated, on the assumption that there is no dynamic symmetry. Necessary and sufficient conditions are derived for the existence of two integrals, each quadratic in the components of the absolute angular velocity of the carrier. It is shown that the initial dynamical system can be reduced to an autonomous gyrostat system if and only if the motion has these two quadratic integrals; the explicit form of a linear transformation to the autonomous system is indicated. The explicit form of the integrals and conditions for their existence are obtained. Examples of motion with two quadratic integrals are considered.

1. A mechanical system K consists of an unchangeable rigid body K_1 (the carrier) and a subsystem K_2 whose elements move in a given manner relative to the carrier. The subsystem K_2 may contain a configurationally variable component and a component of variable composition.

The equations of rotational motion of the carrier of a system of variable composition are well known [1]. In this paper we will derive another form for these equations, which we proposed in [2].

Let E_1 be an inertial frame of reference (FR) let, E_2 be an FR rigidly attached to the carrier, and let E_3 be the principal FR, associated with the principal central axes of inertia of the system K. We introduce the following notation: m_n is the mass of a point mass M_n, C is the centre of mass of K, $\mathbf{r} = \mathbf{CM}_n$; $(\ )_i^\bullet$ denotes the derivative with respect to time in the FR E_i and $(\ )^\bullet = (\ )_2^\bullet$; further

$$\mathbf{G}_i = \sum m_n \mathbf{r}_n \times (\mathbf{r}_n)_i^\bullet, \quad \mathbf{M}_i^f = -\sum m_n \mathbf{r}_n \times (\mathbf{r}_n)_i^{\bullet\bullet} \tag{1.1}$$

The summation in this formula is performed over all the elements of K; $\mathbf{G}_i$ and $\mathbf{M}_i^f$ are the angular momentum and principal moment of inertia (about C) in the motion of the elements of K relative to a FR which is in translational motion, together with C, relative to E_i.

Let J denote the inertia operator of the system K at its centre of mass C. Define a symmetric linear operator Λ by

$$\Lambda\mathbf{x} \equiv J^\bullet\mathbf{x} - \sum m_n[\mathbf{r}_n^\bullet \times (\mathbf{x} \times \mathbf{r}_n) + \mathbf{r}_n \times (\mathbf{x} \times \mathbf{r}_n^\bullet)] \tag{1.2}$$

The right-hand side of this identity remains unchanged if the differentiation in E_2 is replaced throughout by differentiation in any of the FRs E_i.

If $\mathbf{x}_{ij}$ is the angular velocity of E_i relative to E_j, then

$$J\mathbf{x}_{ij} = \mathbf{G}_j - \mathbf{G}_i \tag{1.3}$$

$$\Lambda\mathbf{x}_{ij} = \mathbf{M}_j^f - \mathbf{M}_i^f + (\mathbf{G}_j)_j^\bullet - (\mathbf{G}_i)_i^\bullet \tag{1.4}$$

Suppose the system K is subject to certain external forces with principal moment of inertia $\mathbf{M}^e$ about C; let $\mathbf{M}_r$ denote the principal moment of inertia (about C) of the reactive forces; there may also be a control moment $\mathbf{M}^*$ given in E_2. Let $\mathbf{M} = \mathbf{M}^e + \mathbf{M}_r + \mathbf{M}^*$. Since the passage from the inertial FR E_1 to FR E'_1 moving forward together with C relative to E_1 does not produce Coriolis forces of inertia, while the principal moment (about C) of the translational forces of inertia is zero, it follows that the sum of the moments $\mathbf{M}$ and the principal moment $\mathbf{M}_1^f$ of inertia in motion relative to E'_1 is zero. Hence, using property (1.4) of the operator Λ for $i = 2, j = 1$, we obtain $(\mathbf{G}_1)_1^\bullet = \Lambda\mathbf{x}_{21} + \mathbf{M} + \mathbf{M}_2^f + \overset{\bullet}{\mathbf{G}}_2$. Changing

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here to differentiation in E_2 and taking into account that $\mathbf{G}_1 = \mathbf{G}_2 + J\mathbf{x}_{21}$, we obtain the equation

$$\dot{\mathbf{y}} = \mathbf{y}\times\mathbf{x} + \Lambda\mathbf{x} + \mathbf{L}, \quad \mathbf{y} = J\mathbf{x} + \mathbf{G}_2 \tag{1.5}$$

$$(\mathbf{L} = \mathbf{M} + \mathbf{M}_2^f + \dot{\mathbf{G}}_2, \quad \mathbf{M} = \mathbf{M}^c + \mathbf{M}_r + \mathbf{M}^*)$$

Here and henceforth $\mathbf{x} = \mathbf{x}_{21}$ is the absolute angular velocity of the carrier.

An equation of type (1.5), supplemented by Poisson's equation, also describes the motion of the carrier about a fixed point [3].

In the case considered below—free motion in a uniform field—we have $\mathbf{M}^c = 0$, and Eq. (1.5) for the rotation of the carrier may be separated from the system which describes the motion of the carrier as a whole. We shall assume that the configuration and composition of system K vary according to given laws, in which case the operators $J(t)$, $\Lambda(t)$ and the vector-valued functions $\mathbf{G}_2(t)$ and $\mathbf{L}(t)$ are given in E_2; we shall assume that J, $\mathbf{G}_2 \in C^1[0, +\infty)$, Λ, $\mathbf{L} \in C[0, +\infty)$.

There are several publications† which investigate systems of variable configuration, the reactive forces on which are due to detachment of a working medium whose particles have given absolute velocities. The equations describing the rotation of the carrier for this model are obtained from (1.5) by putting $\Lambda \equiv 0$. For a configurationally variable system of constant composition we also have $\Lambda \equiv 0$.

2. Throughout, we shall assume that the system lacks dynamic symmetry. The conditions for a quadratic integral (QI) to exist for system (1.6) in full form

$$(\mathbf{y}, B\mathbf{y}) + (\mathbf{m}, \mathbf{y}) + \varphi(t) = \text{const} \tag{2.1}$$

obtained previously in [2], are given by Theorem 1.

Notation: A_i and $\mathbf{e}_i$ are the eigenvalues and eigenvectors, respectively, of the inertia operator J, $|\mathbf{e}_i| = 1$, $\lambda_{ij} = (\mathbf{e}_i, \Lambda\mathbf{e}_j)$, $x^{(k)} = (\mathbf{x}, \mathbf{e}_k)$, $\Delta A_i = (A_j - A_k)\delta_{ijk}$, $a_i = A_i\Delta A_i$, $f_i = a_i\beta_{1i}$, where

$$\beta_{1i} = \exp\left(-2\int_0^t A_i^{-1}(\xi)\lambda_{ii}(\xi)d\xi\right), \quad i = 1, 2, 3 \tag{2.2}$$

Theorem 1. Assuming that the operator B is non-singular and the eigenvalues of the operator J are different, an integral (2.1) of system (1.5) exists if and only if

$$(\mathbf{G}_s)_s^{\boldsymbol{\cdot}} + \Lambda\mathbf{x}_{s2} = \mathbf{L} \tag{2.3}$$

and the functions f_i are linearly dependent. The integral may be reduced to the form

$$(J\mathbf{x}_{s1}, B_1C_sJ\mathbf{x}_{s1}) = \text{const} \tag{2.4}$$

The eigenvectors of the operators B_1 and C_s are $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and β_{1i} are the eigenvalues of B_1. The eigenvalues c_{si} of C_s may be any sequence of constants in the linear dependence conditions for the functions f_i

$$c_{s1}f_1 + c_{s2}f_2 + c_{s3}f_3 \equiv 0, \quad c_{si} \equiv \text{const} \tag{2.5}$$

The angular velocity $\mathbf{x}_{s3}$ of the basis E_s about E_3, which appears in condition (2.3), is given by the condition

$$a_k\beta_k x_{s3}^{(k)} = \delta_{ijk}(A_i\beta_i - A_j\beta_j)G_3^{(k)} - (A_i\beta_i + A_j\beta_j)\lambda_{ij} \tag{2.6}$$

where (i, j, k) is any permutation of (1, 2, 3), $\beta_i = \beta_{1i}c_{si}$. The necessary condition (2.3) may be written as follows [2]:

$$\mathbf{M} + \mathbf{M}_s^f = 0 \tag{2.7}$$

where $\mathbf{M} = \mathbf{M}r + \mathbf{M}^*$, and, noting that $\mathbf{M} + \mathbf{M}_1^f = 0$, we obtain $\mathbf{M}_s^f = \mathbf{M}_1^f$.

†MAKEYEV, N. N., Integral manifolds of the equations of dynamics for compound mechanical systems. Doctorate dissertation: 01.02.01. St Petersburg, 1992.

Definition 1. A motion of the FR E_s with pole at the centre of mass C of K is said to be quasi-translational if the principal moment of inertia (about C) in motion relative to E_s is equal to the principal moment of inertia (about C) relative to the FR E_1', that is, $\mathbf{M}_s^f = \mathbf{M}_1^f$. The frame of reference E_s and its basis will be called a QT-frame and a QT-basis, respectively.

Here E_1', as before, is a frame of reference in translational motion, together with C, with respect to the inertial FR E_1. It is obvious that E_1' is a QT-frame of reference.

The necessary condition (2.3) for the existence of the integral means that the basis E_s is a QT-basis.

Condition (2.5) that the functions f_i be linearly dependent may be written as $n_f < 3$, where n_f is the number of dimensions of the linear span of the set $\{f_1, f_2, f_3\}$.

By (2.6), any sequence of constants c_{s1}, c_{s2}, c_{s3} uniquely defines $\mathbf{x}_{s3}$. The following definition, which assumes that $n_f < 3$, is introduced for brevity.

Definition 2. A basis E_s whose angular velocity $\mathbf{x}_{s3}$ about the principal basis is given by (2.6), where the constant eigenvalues c_{si} of C_s satisfy condition (2.5), will be called an admissible basis corresponding to the operator C_s. A FR with pole at C and an admissible basis will be called an admissible FR.

Theorem 1 may now be rephrased as follows.

Theorem 1′. Assuming that the operator B is non-singular and the eigenvalues of the operator J are different, a QI (2.1) of system (1.5) exists if and only if $n_f < 3$ and an admissible FR corresponding to some operator C_s is a QT-frame. The integral may be written in the form (2.4).

If $n_f = 2$, condition (2.5) defines the sequence of constants c_{si} uniquely (apart from a common multiple) and the system may have only one QI (2.1), which is written as (2.4). A necessary condition for the existence of two independent QIs is $n_f = 1$, i.e.

$$f_i = \beta_{1i} a_i = a_{i0} f(t), \quad f(0) = 1; \quad i = 1, 2, 3 \ (a_{i0} = a_i(0)) \tag{2.8}$$

The conditions for the existence of two independent QIs, that follow from Theorem 1′, may be formulated as follows.

Proposition 1. If system (1.5) lacks dynamic symmetry, it has two independent QIs in full form if and only if the functions $f_i(t)$ are proportional to one another and two admissible QT-bases exist corresponding to different operators C_n and C_m. The integrals are then

$$(J\mathbf{x}_{n1},\ B_1 C_n J \mathbf{x}_{n1}) = \text{const}, \quad (J\mathbf{x}_{m1},\ B_1 C_m J \mathbf{x}_{m1}) = \text{const} \tag{2.9}$$

Later we shall present a simpler system of conditions for the existence of two QIs.

3. We will now describe the set of admissible bases in the case when two independent QIs exist.

Assume, indeed, that two independent QIs exist. Then they may be written in the form (2.9). The constant eigenvalues of the operators C_n and C_m must satisfy condition (2.5), which, in view of (2.8), implies that the triples $\{c_{ni}\}$, $\{c_{mi}\}$ are linearly independent solutions of the equation

$$a_{10} u_1 + a_{20} u_2 + a_{30} u_3 = 0$$

Now, $u_i = 1$ and $u_i = A_{i0}^{-1}$ $(i = 1, 2, 3)$ are also independent solutions of this equation. Since any linear combination of integrals (2.9) is again a QI, this implies the following.

Proposition 2. If system (1.5) lacks dynamic symmetry and has two independent QIs in full form, then any QI of the system is a linear combination of the integrals

$$(J\mathbf{x}_{41},\ B_1 J \mathbf{x}_{41}) = \text{const}, \quad (J\mathbf{x}_{51},\ B_1 J_0^{-1} J \mathbf{x}_{51}) = \text{const} \tag{3.1}$$

where E_4 is an admissible basis corresponding to the identity operator E and E_5 is an admissible basis corresponding to J_0^{-1}, $J_0 = J(0)$.

Proposition 3. If two QIs exist, then the angular velocity about E_5 of an arbitrary admissible basis E_s is

$$\mathbf{x}_{s5} = (J_0 + \nu_s E)^{-1} J_0 \mathbf{x}_{45}, \quad \nu_s = \text{const} \tag{3.2}$$

Proof. By Proposition 2, any QI may be written as a linear combination of integrals (3.1). On the other hand, by Theorem 1, any QI has the form (2.4). Hence

$$\alpha_{s1}(J\mathbf{x}_{41}, B_1 J\mathbf{x}_{41}) + \alpha_{s2}(J\mathbf{x}_{51}, B_1 J_0^{-1} J\mathbf{x}_{51}) = (J\mathbf{x}_{s1}, B_1 C_s J\mathbf{x}_{s1}) + \text{const}$$

where $\mathbf{x}_{k1} = \mathbf{x}_{k2} + \mathbf{x}$, $\mathbf{x} = \mathbf{x}_{21}$. Comparison of terms quadratic and linear in $\mathbf{x}$ yields the conditions

$$C_s = \alpha_{s1}E + \alpha_{s2}J_0^{-1}, \quad \alpha_{s1}\mathbf{x}_{42} + \alpha_{s2}J_0^{-1}\mathbf{x}_{52} = (\alpha_{s1}E + \alpha_{s2}J_0^{-1})\mathbf{x}_{s2}$$

It follows from this condition that $(\alpha_{s1}E + \alpha_{s2}J_0^{-1})\mathbf{x}_{s5} = \alpha_{s1}\mathbf{x}_{45}$, and, putting $\nu_s = \alpha_{s2}\alpha^{-1}_{s1}$, we obtain a representation of the whole set of admissible angular velocities in the form (3.2), where ν_s is an arbitrary constant parameter. For an admissible basis E_5 we have $\mathbf{x}_{s5} = 0$, and in this case we can retain representation (3.2) by setting $\nu_5 = \infty$.

It follows from formula (2.6), in view of (2.8), that

$$\begin{aligned} x_{43}^{(k)} &= a_{k0}^{-1}[(A_{i0}h_i - A_{j0}h_j)G_3^{(k)}\delta_{ijk} - (A_{i0}h_i + A_{j0}h_j)\lambda_{ij}] \\ x_{53}^{(k)} &= A_{k0}a_{k0}^{-1}[(h_i - h_j)G_3^{(k)}\delta_{ijk} - (h_i + h_j)\lambda_{ij}] \\ x_{54}^{(k)} &= h_i h_j a_{k0}^{-1}[(\Delta A_i - \Delta A_j)\delta_{ijk}\lambda_{ij} - \Delta A_k G_3^{(k)}] \end{aligned} \tag{3.3}$$

where $A_{i0} = A_i(0)$, $h_i = \Delta A_{i0}(\Delta A_i)^{-1}$.

The case $\mathbf{x}_{45} = 0$ (the bases E and E_5 are identical) was dealt with in [2].

We will now present another description of the set of admissible angular velocities, equivalent to (3.2).

Let us denote an operator with eigenvalues and eigenvectors λ_{ii}, $\mathbf{e}_i (i = 1, 2, 3)$ by Λ_1 [2], and put $\Lambda_2 = \Lambda - \Lambda_1$. It follows from (2.2) that

$$(B_1)_3^{\cdot} = -2B_1 J^{-1}\Lambda_1 \tag{3.4}$$

Condition (2.6) is satisfied if and only if the operator F_1 is skew symmetric [2]

$$F_1 = B_1 C_s F J^{-1}, \quad F = \Lambda_2 + M(\mathbf{G}_3 - J\mathbf{x}_{s3}) + M(\mathbf{x}_{s3})J \tag{3.5}$$

Throughout, $M(\mathbf{b})$ is the operator of vector multiplication by $\mathbf{b}$: $M(\mathbf{b})\mathbf{x} \equiv \mathbf{b} \times \mathbf{x}$.

We make the change of variables $\mathbf{x}_{s3} = \mathbf{w} + \mathbf{a}$, choosing $\mathbf{a}$ so that the operator $\Lambda_2 + M(\mathbf{G}_3 - J\mathbf{a}) + M(\mathbf{a})J$ is skew symmetric with associated vector $\mathbf{c}$

$$\Lambda_2 + M(\mathbf{G}_3 - J\mathbf{a}) + M(\mathbf{a})J = M(\mathbf{c}) \tag{3.6}$$

This equality determines the vectors $\mathbf{a}$ and $\mathbf{c}$. Since Λ_2 is a symmetric operator, it follows from (3.6) that

$$\Lambda_2 = (JM(\mathbf{a}) - M(\mathbf{a})J)/2, \quad M(\mathbf{c}) = M(\mathbf{G}_3 - J\mathbf{a}) + (JM(\mathbf{a}) + M(\mathbf{a})J)/2 \tag{3.7}$$

$$a^{(k)} = -2\lambda_{ij}(\Delta A_k)^{-1}, \quad c^{(k)} = G_3^{(k)} + (\Delta A_j - \Delta A_i)(\Delta A_k)^{-1}\lambda_{ij}\delta_{ijk} \tag{3.8}$$

The operator F takes the form $F = M(\mathbf{c}) + M(\mathbf{w})J - M(J\mathbf{w})$.

Putting $D = B_1 C_s$, we write the condition $F_1 = -F_1^{\mathrm{T}}$ in the form

$$DM(\mathbf{c} - J\mathbf{w})J^{-1} - J^{-1}M(\mathbf{c} - J\mathbf{w})D = M(\mathbf{w})D - DM(\mathbf{w})$$

Multiplying by $\mathbf{e}_i$ and $\mathbf{e}_j$, we obtain the equivalent system of conditions

$$(\mathbf{c} - J\mathbf{w})^{(k)}(d_j A_i^{-1} - d_i A_j^{-1}) = w^{(k)}(d_i - d_j) \tag{3.9}$$

where d_i are the eigenvalues of D and (i, j, k) is a permutation of $(1, 2, 3)$.

Since $C_s = \alpha_{s1}E + \alpha_{s2}J_0^{-1}$, it follows in view of (2.8) that

$$d_i = a_{i0}a_i^{-1}(\alpha_{s1} + \alpha_{s2}A_{i0}^{-1})f$$

The system of conditions (3.9) becomes

$$J\mathbf{w} + \lambda_s \mathbf{w} = \mathbf{c} \tag{3.10}$$

where

$$\lambda_s(t) = A_1 A_2 A_3 (\sum A_i (A_{i0} + \nu_s)\Delta A_{i0})^{-1} \sum A_i^{-1}(A_{i0} + \nu_s)\Delta A_{i0} \tag{3.11}$$

Here and below, the summation over i is from $i = 1$ to $i = 3$; $\nu_s = \alpha_{s1}\alpha_{s2}^{-1}$.
We have thus proved the following proposition.

Proposition 4. If two QIs exist, then the angular velocity about E_3 of an arbitrary admissible FR E_s may be written as

$$\mathbf{x}_{s3} = \mathbf{a} + (J + \lambda_s E)^{-1}\mathbf{c} \tag{3.12}$$

Proposition 5. The descriptions (3.2) and (3.12) of the set of admissible angular velocities are equivalent.

Proof. It follows from (3.12) that

$$\mathbf{x}_{45} = (\mathbf{x}_{43} - \mathbf{a}) - (\mathbf{x}_{53} - \mathbf{a}) = (J + \lambda_4 E)^{-1}[\mathbf{c} - (J + \lambda_4 E)(\mathbf{x}_{53} - \mathbf{a})]$$

and formula (3.12) may be written in the form

$$\mathbf{x}_{s5} = (J + \lambda_s E)^{-1}[(\lambda_4 - \lambda_s)(\mathbf{x}_{53} - a) + (J + \lambda_4 E)\mathbf{x}_{45}]$$

which is identical with (3.2) if

$$[(\lambda_s - \lambda_4)J_0 + \nu_s(J + \lambda_4 E)]\mathbf{x}_{45} = (\lambda_4 - \lambda_s)(J_0 + \nu_s E)(\mathbf{x}_{53} - \mathbf{a})$$

It can now be verified directly that this equality is an identity in ν_s, using the explicit expressions (3.3), (3.8) and (3.11) for $\mathbf{x}_{53}$, $\mathbf{x}_{54}$, $\mathbf{a}$, λ.

Proposition 6. The set of admissible bases of a system with two QIs has the invariant $\mathbf{G}_s \times \mathbf{x}_{s3} + \Lambda_2 \mathbf{x}_{s3}$, that is, the following equality holds

$$\mathbf{G}_s \times \mathbf{x}_{s3} + \Lambda_2 \mathbf{x}_{s3} = \mathbf{p}, \quad \mathbf{p} = (J\mathbf{a} + \mathbf{c}) \times \mathbf{a} \tag{3.13}$$

where $\mathbf{p}$ is independent of the choice of the admissible basis E_s.

Proof. Condition (3.10), defining the set of admissible bases, implies that

$$0 = (J\mathbf{w} - \mathbf{c}) \times \mathbf{w} = (J\mathbf{x}_{s3} - J\mathbf{a} - \mathbf{c}) \times (\mathbf{x}_{s3} - \mathbf{a})$$

whence

$$(J\mathbf{x}_{s3}) \times \mathbf{x}_{s3} = (J\mathbf{a} + \mathbf{c}) \times \mathbf{x}_{s3} + (J\mathbf{x}_{s3}) \times \mathbf{a} - (J\mathbf{a} + \mathbf{c}) \times \mathbf{a}$$

Taking (3.6) into consideration, we obtain (3.13).

4. We will note some properties of admissible QT-bases.

Proposition 7. For any admissible QT-basis E_s of a system with two QIs, the following equation holds

$$B_1^{-1/2}(B_1^{1/2} J\mathbf{x}_{s3})_3^{\bullet} = \mathbf{b}, \quad \mathbf{b} = \mathbf{p} - \mathbf{M}_3^f - \mathbf{M} \tag{4.1}$$

Proof. Let E_s be some admissible QT-basis. Then condition (2.7) is satisfied and consequently so is condition (2.3); changing to differentiation in E_3, we can write the latter in the form

$$(\mathbf{G}_3 - J\mathbf{x}_{s3})_3^{\bullet} + \mathbf{G}_s \times \mathbf{x}_{s3} + \Lambda \mathbf{x}_{s2} = \mathbf{L}$$

Hence, assuming that E_s is an admissible basis and taking property (3.13) into account, we obtain

$$(J\mathbf{x}_{s3})_3^{\bullet} - \Lambda_1\mathbf{x}_{s3} = \mathbf{p} + \Lambda\mathbf{x}_{32} + (\mathbf{G}_3)_3^{\bullet} - \mathbf{L} \tag{4.2}$$

It follows from formula (3.4) that $(J\mathbf{x})_3^{\bullet} - \Lambda_1\mathbf{x} = B_1^{-1/2}(B_1^{1/2}J\mathbf{x})_3^{\bullet}$. Using property (1.4) of the operator Λ, we conclude that the right-hand side of (4.2) is equal to $\mathbf{b}$, and this proves (4.1).

Proposition 8. For a system with two QIs, the angular velocity $\mathbf{x}_{mn}$ of rotation of two arbitrary admissible QT-bases about one another satisfies the condition

$$(B_1^{1/2}J\mathbf{x}_{mn})_3^{\bullet} = 0 \tag{4.3}$$

Proof. This follows from Proposition 7.

Proposition 9. If a system has two QIs, then condition (4.3) holds for all admissible bases if and only if it holds for two different admissible bases. In particular, an equivalent form of condition (4.3) is

$$B_1^{1/2}J\mathbf{x}_{45} = \mathbf{G}_0,\ (\mathbf{G}_0)_3^{\bullet} = 0 \tag{4.4}$$

Proof. It follows from (3.2) that

$$\mathbf{x}_{mn} = (J_0 + \nu_m E)^{-1}(J_0 + \nu_n E)^{-1}(\nu_n - \nu_m)J_0\mathbf{x}_{45}$$

and it is clear that (4.3) and (4.4) are equivalent.

5. We will now show that the necessary conditions developed above are also sufficient for the existence of two QIs.

Theorem 2. If system (1.5) describing the rotation of the carrier of a variable system in free motion lacks dynamic symmetry, it has two QIs in full form if and only if it satisfies conditions (2.8), (4.4) and the condition

$$B_1^{-1/2}(B_1^{1/2}J\mathbf{x}_{43})_3^{\bullet} = \mathbf{b} \tag{5.1}$$

where E_4 and E_5 are admissible bases corresponding to the operators E and J_0^{-1}.

Proof. The necessity of conditions (2.8), (4.4) and (5.1) was demonstrated above; we will prove that they are also sufficient. We claim that if condition (5.1) is satisfied, then the admissible basis E_4 is a QT-basis. In view of (3.4), we can write this condition as $(J\mathbf{x}_{43})_3^{\bullet} - \Lambda_1\mathbf{x}_{43} = \mathbf{b}$. Using property (3.13) of admissible bases and formula (1.4), we can write condition (5.1) as

$$(J\mathbf{x}_{43})_3^{\bullet} = \mathbf{G}_4 \times \mathbf{x}_{43} - \mathbf{M}_4^f - \mathbf{M} + (\mathbf{G}_3)_3^{\bullet} - (\mathbf{G}_4)_4^{\bullet}$$

Hence, by (1.3), we have $\mathbf{M} + \mathbf{M}_4^f = (\mathbf{G}_3 - \mathbf{G}_4 - J\mathbf{x}_{43})_3^{\bullet} \equiv 0$, and E_4 is a QT-basis.

If conditions (4.4) and (5.1) are satisfied, then $B_1^{-1/2}(B_1^{1/2}J\mathbf{x}_{53})_3^{\bullet} = \mathbf{b}$ and the admissible basis E_5 is also a QT-basis. By Proposition 1, the initial system has two QIs.

When $\Lambda \equiv 0$, the criterion for the existence of two QIs is as follows.

Theorem 3. If $\Lambda \equiv 0$ and the system lacks dynamic symmetry, system (1.5) has two independent QIs in full form if and only if the following conditions hold

1) $a_i = a_{i0}f(t);\ f(0) = 1,\ i = 1,2,3$

2) $\mathbf{G}_3 = (A_1A_2A_3)^{-1}f(t)J\mathbf{k}_0,\ (\mathbf{k}_0)_3^{\bullet} = 0$

3) $\mathbf{L} = 0.$

Proof. When $\Lambda \equiv 0$ we have $B_1 = E$, and Condition 1 follows from condition (2.8). From (3.3) we obtain

$$\mathbf{x}_{43} = J^{-1}\mathbf{G}_3, \quad \mathbf{x}_{45} = A_1A_2A_3(A_{10}A_{20}A_{30}f)^{-1}J_0J^{-2}\mathbf{G}_3 \tag{5.2}$$

Condition (5.1) yields Condition 2 of our theorem.

Condition (5.1) may be written as $(\mathbf{G}_3)_3^\bullet = \mathbf{b}$. When $\Lambda \equiv 0$ we have $\mathbf{a} = \mathbf{p} = 0$, $\mathbf{c} = \mathbf{G}_3$ and $\mathbf{b} = -\mathbf{M}_3^f - \mathbf{M}$. It follows from (1.4) that

$$\mathbf{M}_3^f + (\mathbf{G}_3)_3^\bullet = \mathbf{M}_2^f + (\mathbf{G}_2)_2^\bullet$$

Then $\mathbf{b} = (\mathbf{G}_3)_3^\bullet - \mathbf{L}$, that is, $\mathbf{L} = 0$, which completes the proof.

In Condition 1 of Theorem 3, only two of the three equalities can be independent; this condition defines a circular cone in the space of the variables $\eta_i = a_{i0}^{-1}A_i$ $(i = 1, 2, 3)$

$$\eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3 = 0 \tag{5.3}$$

Consequently, Condition 1 imposes only one restriction on the admissible laws governing the variation of the principal moments of inertia $A_i(t)$.

For a variable system of constant composition $\mathbf{M}_r = 0$, and if there is no additional control moment $\mathbf{M}^*$ we have $\mathbf{L} = 0$.

Proposition 10. The free motion of a configurationally variable system of constant composition, lacking dynamic symmetry, in a uniform gravitational field has two independent QIs in full form if and only if Conditions 1 and 2 of Theorem 3 are satisfied.

6. We will now show that, if two independent QIs exist, the initial system (1.5) can be reduced to an autonomous gyrostat system (see below, formula (6.7)). Since this autonomous system has two QIs (formula (6.9)), the conditions of Theorem 2 are necessary and sufficient for system (1.5) to be reducible to an autonomous gyrostat system.

Proposition 11. The condition $n_f = 1$ is equivalent to the identity

$$B_1J(\mathbf{x}\times J\mathbf{x}) \equiv f(t)J_0(\mathbf{x}\times J_0\mathbf{x}) \tag{6.1}$$

Proof. It can be verified directly that

$$J\mathbf{x}\times\mathbf{x} \equiv \Delta A_1x^{(2)}x^{(3)}\mathbf{e}_1 + \Delta A_2x^{(1)}x^{(3)}\mathbf{e}_2 + \Delta A_3x^{(1)}x^{(2)}\mathbf{e}_3$$

and this implies that (6.1) is equivalent to conditions (2.8).

Now suppose that the admissible bases E_4 and E_5, defined in Proposition 2, are QT-bases. Then condition (2.3) holds for $s = 4$ and $s = 5$, and system (1.5) may be written

$$\Lambda\mathbf{x}_{51} = (\mathbf{G}_1)_1^\bullet - (\mathbf{G}_5)_5^\bullet$$

Put $\mathbf{v} = \mathbf{x}_{51}$; taking (1.3) into consideration, we obtain

$$(J\mathbf{v})_3^\bullet + \mathbf{v}\times J\mathbf{v} = \Lambda_1\mathbf{v} + \Lambda_2\mathbf{v} + \mathbf{x}_{53}\times J\mathbf{v} + (\mathbf{G}_3 - J\mathbf{x}_{53})\times\mathbf{v} \tag{6.2}$$

Substituting Λ_2 into this equality from (3.6) and putting $\mathbf{w}_5 = \mathbf{x}_{53} - \mathbf{a}$, we conclude, using (3.10), that $\mathbf{w}_5 = (\lambda_5E + J)^{-1}\mathbf{c}$, and Eq. (6.2) becomes

$$(J\mathbf{v})_3^\bullet + \mathbf{v}\times J\mathbf{v} = \Lambda_1\mathbf{v} + [(J+\lambda_5E)^{-1}\mathbf{c}]\times[(J+\lambda_5E)\mathbf{v}] \tag{6.3}$$

Proposition 12. If $n_f = 1$, then

$$(J+\lambda_5E)^{-1} = \nu B_1^{-1}J^{-1}J_0, \quad \nu = -(\Delta A_1\Delta A_2\Delta A_3)^{-1}f(t)\Sigma A_i\Delta A_{i0} \tag{6.4}$$

Proof. It was observed in the proof of Proposition 3 that formula (3.2) can be used for the basis E_5 with $\nu_5 = \infty$. Then, using (3.11), we have

$$(A_i+\lambda_5)^{-1} = -(\Delta A_1\Delta A_2\Delta A_3\Delta A_{i0})^{-1}\Delta A_i\Sigma A_i\Delta A_{i0}$$

and, taking condition (2.8) into account, we obtain formula (6.4).

The right-hand side of Eq. (6.3) may be transformed, using (6.4), to the form $\Lambda_1\mathbf{v} + (B_1^{-1}J^{-1}J_0\mathbf{c}) \times (B_1JJ_0^{-1}\mathbf{v})$. Comparing formulae (3.3) and (3.8) and taking (2.8) into consideration, we obtain the relations

$$\mathbf{c} = \Pi_{-1}f(t)J_0^{-1}J^2B_1^2\mathbf{x}_{54}, \quad \Pi_\lambda = \prod_{i=1}^{3} A_{i0}A_i^{-1}\beta_{1i}^\lambda \tag{6.5}$$

We introduce the variable

$$\mathbf{u} = B_1^{1/2}J_0^{-1}J\mathbf{v} = B_1^{1/2}J_0^{-1}J\mathbf{x}_{51} \tag{6.6}$$

Taking (6.5) and (4.4) into consideration, we can rewrite Eq. (6.3) as

$$J_0(B_1^{-\frac{1}{2}}\mathbf{u})_3^\bullet + (J^{-1}J_0B_1^{-\frac{1}{2}}\mathbf{u}) \times (J_0B_1^{-\frac{1}{2}}\mathbf{u}) = \Pi_{-1}f(t)(B_1^{\frac{1}{2}}\mathbf{G}_0) \times (B_1^{\frac{1}{2}}\mathbf{u}) + \Lambda_1J^{-1}J_0B_1^{-\frac{1}{2}}\mathbf{u}$$

Since a symmetric operator A with eigenvalues a_1, a_2 and a_3 satisfies the identity $A(A\mathbf{b}_1 \times A\mathbf{b}_2) \equiv a_1a_2a_3\mathbf{b}_1 \times \mathbf{b}_2$, it follows, in view of formula (3.4) and identity (6.1), that the above differential equation for $\mathbf{u}$ can be expressed in the form

$$J_0(\mathbf{u})_3^\bullet = (J_0\mathbf{u} + \mathbf{G}_0) \times \mathbf{u}f(t)\Pi_{-1/2}$$

Changing to the variable τ, $d\tau = \Pi_{-1/2}f(t)dt$, we obtain an autonomous system

$$J_0\frac{d\mathbf{u}}{d\tau}\Big|_{E_3} = (J_0\mathbf{u} + \mathbf{G}_0) \times \mathbf{u} \tag{6.7}$$

In view of (2.8), the formula defining $\tau(t)$ may be rewritten as

$$d\tau = \left(f^{-1}\prod_{i=1}^{3}\frac{A_{i0}\Delta A_i}{\Delta A_{i0}A_i}\right)^{1/2}dt \tag{6.8}$$

System (6.7) is identical with Euler's system for the absolute angular velocity of a gyrostat with constant gyrostatic moment. This system can be integrated in quadratures and it has two QIs

$$|J_0\mathbf{u} + \mathbf{G}_0| = \text{const}, \quad (\mathbf{u}, J_0\mathbf{u}) = \text{const} \tag{6.9}$$

We have proved the following theorem.

Theorem 4. If system (1.5), describing the rotation of the carrier in the free motion of a variable system in a uniform gravitational field, lacks dynamic symmetry, it is reducible by a non-singular linear transformation $\mathbf{u} = A\mathbf{x} + \mathbf{d}$, $\tau = \tau(t)$, to the autonomous dynamical system of a gyrostat (6.7) if and only if the initial system (1.5) has two QIs in full form.

7. We will show that a simple relation exists between the rotations relative to the carrier of the admissible QT-bases and the uniform rotations of the equivalent gyrostat, provided the system has two QIs.

System (6.7) has a class of solutions of the form $\mathbf{u}$ = const in E_3 which correspond to the uniform rotations of the equivalent gyrostat. In this case $(J_0\mathbf{u} + \mathbf{G}_0)\|\mathbf{u}$ and

$$\gamma_1(J_0\mathbf{u} + \mathbf{G}_0) + \gamma_2\mathbf{u} = 0, \quad \gamma_1, \gamma_2 = \text{const} \tag{7.1}$$

This condition defines the entire set of absolute angular velocities of uniform rotations of the gyrostat.

If the initial system (1.5) has two QIs, then any of its solutions may be obtained using transformation (6.6) from some solution of the autonomous system (6.7). The set of particular solutions of system (1.5) that correspond to uniform rotations of the equivalent gyrostat is defined by a condition obtained from condition (7.1) by taking note of relation (6.6), where $\mathbf{x}_{51} = \mathbf{x} + \mathbf{x}_{52}$

$$\gamma_1J_0B_1^{1/2}J_0^{-1}J(\mathbf{x} + \mathbf{x}_{52}) + \gamma_1\mathbf{G}_0 + \gamma_2B_1^{1/2}J_0^{-1}J(\mathbf{x} + \mathbf{x}_{52}) = 0$$

Substituting $\mathbf{G}_0$ into this relation from condition (4.4), we obtain

$$(\gamma_1 J_0 + \gamma_2 E)(\mathbf{x} + \mathbf{x}_{52}) + \gamma_1 J_0 \mathbf{x}_{45} = 0 \tag{7.2}$$

By Proposition 3, the angular velocity $\mathbf{x}_{2s}$ of the carrier relative to some admissible basis E_s may be written as

$$\mathbf{x}_{2s} = \mathbf{x}_{25} + \mathbf{x}_{5s} = \mathbf{x}_{25} - (J_0 + \nu_s E)^{-1} J_0 \mathbf{x}_{45}$$

Consequently, if $\nu_s = \gamma_2\gamma_1^{-1}$, the vector $\mathbf{x} = \mathbf{x}_{2s}$ satisfies condition (7.2). We have established the following property.

Proposition 13. The angular velocity of the carrier of a variable system with two QIs relative to any of the admissible QT-bases of the system is a particular solution of system (1.5), which corresponds (via formula (6.6)) to the absolute angular velocity of some uniform rotation of the equivalent gyrostat.

8. Let us consider some examples of the motion of variable systems with two QIs.

Example 1. Let K be a configurationally variable system of constant composition whose centre of mass C and principal axes remain fixed relative to the carrier, and whose angular momentum (about C) of motion relative to the carrier is zero. We also assume that there is no dynamic symmetry.

Let us apply Theorem 3. Condition 3 is satisfied. The bases E_3 and E_2 are identical $\mathbf{G}_3 = 0$ and Condition 2 is also satisfied, with $\mathbf{k}_0 = \mathbf{G}_0 = 0$.

The system K can be implemented as a system consisting of three pairs of point masses on three mutually perpendicular axes. Each pair is symmetrically positioned with respect to the point C. The distance to the point of intersection C of the axes is equal to ρ_i, and the mass of each point of a pair is equal to m_i, respectively. The basis E_2 is associated with the given axes.

By Theorem 3, system K will have two QIs if the distances $\rho_i(t)$ satisfy the condition obtained from (5.3) in this case

$$\xi_1^4 - \xi_2^4 + \xi_3^4 = 0;\; \xi_i = m_i^{1/2}\chi_i^{1/4}\rho_i,\; \chi_i = \left|m_j^2\rho_{j0}^4 - m_k^2\rho_{k0}^4\right| \tag{8.1}$$

Let us assume that $m_1\rho_{10}^2 > m_2\rho_{20}^2 > m_3\rho_{30}^2$.

The quadratic integrals may be written as

$$|J_0^{-1/2} J\mathbf{x}_{21}| = \text{const}, \quad |J\mathbf{x}_{21}| = \text{const} \tag{8.2}$$

Example 2. Suppose the angular momentum of motions in system K, described in Example 1, relative to the carrier is zero. Condition 3 of Theorem 3 holds identically, while Condition 1 implies that the principal moments of inertia satisfy relation (5.3). We can assume, say, that the system consists, as before, of three pairs of displaced point masses, while there is a flywheel at the centre of mass C, whose influence on the moments of inertia of the system will be ignored. Then, as in Example 1, the distances $\rho_i(t)$ must satisfy condition (8.1).

Let us consider the case when the angular momentum $\mathbf{G}_2$ maintains its direction in the principal basis (the axis of rotation of the flywheel is fixed relative to the carrier). Condition 2 of Theorem 3 will obviously hold if $J = \mu(t)J_0$. But if the inertia operator varies in some other way, this condition will hold only if the direction of the angular momentum $\mathbf{G}_2$ of the internal motions, which is fixed in E_2, coincides with one of the principal directions.

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